Midterm 1

Question 1. N students are enrolled in a probability class. The professor forms a team of k students to participate in a mathematics competition.

(i) How many different teams can be formed? [2]

A new student joins the class.

(ii) How many different teams can now be formed? [2]
(iii) How many teams which include the new student can be formed? [2]
(iv) How many teams which do not include the new student can be formed? [2]

Deduce Pascal's identity [2]

$$\binom{N+1}{k} = \binom{N}{k-1} + \binom{N}{k}.$$

Question 2. Every morning Alice tosses a biased coin (giving head with probability p). If it comes up head, she wears a red shirt. If it comes up tail, she tosses a fair coin (giving head with probability $1/2$): if it comes up head she wears a red shirt, and otherwise she wears a white shirt.

(i) What is the probability that today Alice wears a white shirt? [4]
(ii) Given that Alice is wearing a red shirt, what is the probability that the first coin has come up head? [4]

Question 3. Let X_1, X_2, X_3 be independent Bernoulli random variables of parameter $1/2$. Define the additional random variables

$$Y_1 = 2X_1 - 1, \qquad Y_2 = 2X_2 - 1, \qquad Z_1 = Y_1 X_3, \qquad Z_2 = Y_2 X_3.$$

(i) Are Y_1, Y_2 independent? Are Y_1, X_3 independent? Are Y_2, X_3 independent? [3]
(ii) Compute the expectation of Y_1, Y_2, Z_1, Z_2. [2]
(iii) Compute $\mathrm{Cov}(Z_1, Z_2)$. [2]
(iv) Compute $\mathbb{P}(Z_1 = 0, Z_2 = 0)$ and $\mathbb{P}(Z_1 = 0)$, $\mathbb{P}(Z_2 = 0)$. [2]
(v) Are Z_1 and Z_2 independent? Explain. [2]

Question 4. Let X be a Poisson random variable with parameter $\lambda \in (0, \infty)$.

(i) Compute the associated probability generating function $G_X(t) = \mathbb{E}(t^X)$. [3]
(ii) By differentiating G_X or otherwise, compute $\mathbb{E}(X)$. [3]

[You may use that $\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$. If needed, you may exchange derivative and infinite sum.]